SYMMETRY OF n-MODE POSITIVE SOLUTIONS FOR TWO-DIMENSIONAL HÉNON TYPE SYSTEMS

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ABSTRACT. We provide a symmetry result for n-mode positive solutions of a general class of semi-linear elliptic systems under cooperative conditions on the nonlinearities. Moreover, we apply the result to a class of Hénon systems and provide the existence of multiple n-mode positive solutions.

1. Introduction and results

Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ and, for $m \in \mathbb{N}$ with $m \geq 2$, consider the system

$$\begin{cases} \Delta u^i + f^i(|x|, u^i) = 0 & \text{in } D, \\ u^i = 0 & \text{on } \partial D, \end{cases} \qquad \text{for } i = 1, \dots, m, \tag{1.1}$$

where f^i are smooth functions over $(0,1) \times (0,\infty)^m$. Semi-linear elliptic systems as (1.1) arise naturally in many physical and biological contests, see e.g. [8,15,16,18,20] and the references therein. As far as the symmetry of positive solutions is concerned and the functions f^i are decreasing in the radial variable, the celebrated moving plane method [9] can be applied when the system is cooperative namely $\partial f^i/\partial u^j \geq 0$ for every $i \neq j$ [6,13,21]. The aim of this note is to establish a general symmetry result (Theorem 1.1) for n-mode ($2\pi/n$-rotation invariant) solutions, namely solutions $(u^1, \dots, u^m)$ such that each component $u^i : \overline{D} \to \mathbb{R}$, in polar coordinates, satisfies

$$u^i(r,\theta) = u^i(r, \theta + 2\pi/n), \qquad \text{for all } (r,\theta) \in [0,1] \times \mathbb{R},$$

as well as provide a meaningful application of it (Theorem 1.2) to the system of Hénon type

$$\begin{cases} \Delta u + \dfrac{2p}{p+q}|x|^\alpha u^{p-1} v^q = 0 & \text{in } D, \\ \Delta v + \dfrac{2q}{p+q}|x|^\alpha u^p v^{q-1} = 0 & \text{in } D, \\ u > 0, \ v > 0 & \text{in } D, \\ u = v = 0 & \text{on } \partial D. \end{cases} \tag{1.2}$$

Quite recently, these systems were carefully investigated in [22,23] (see also [1,10,11] and references therein) and they can be considered as a vectorial counterpart of the celebrated equation

$$\begin{cases} \Delta u + |x|^\alpha u^{p-1} = 0 & \text{in } D, \\ u > 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

first studied in [17] after being introduced by Hénon in [12] in connection with the research of rotating stellar structures. We shall say that u is of class C^n at the origin if u is of class C^{n-1} in

2000 *Mathematics Subject Classification.* 35B06, 35B09, 35J15.

Key words and phrases. Radial symmetry, Hénon type systems, n-mode positive solutions.

The first author was partially supported by the Grant-in-Aid for Scientific Research (C) (No. 21540214) from Japan Society for the Promotion of Science. The second author was partially supported by 2009 italian MIUR project: "Variational and Topological Methods in the Study of Nonlinear Phenomena".

a neighborhood of the origin and each $(n-1)$-th partial derivative is totally differentiable at the origin. Then we prove the following

Theorem 1.1. *Let $m, n \in \mathbb{N}$ with $m, n \geq 2$ and $f^1, \dots, f^m \in C((0,1) \times (0,\infty)^m, \mathbb{R})$ such that*

(i) for each $i \in \{1, \dots, m\}$ and $(u^1, \dots, u^m) \in (0,\infty)^m$, the map

$$r \mapsto r^{2-2n} f^i(r, u^1, \dots, u^m) : (0,1) \to \mathbb{R}$$

is nonincreasing;

(ii) for each $i \in \{1, \dots, m\}$ and $r \in (0,1)$, $f^i(r, \cdot, \dots, \cdot) \in C^1((0,\infty)^m, \mathbb{R})$;

(iii) for each $i, j \in \{1, \dots, m\}$ with $i \neq j$ and $(r, u^1, \dots, u^m) \in (0,1) \times (0,\infty)^m$,

$$\frac{\partial f^i}{\partial u^j}(r, u^1, \dots, u^m) \geq 0;$$

(iv) for each $i, j \in \{1, \dots, m\}$, $r_0 \in (0,1)$ and $M \in (0,\infty)$,

$$\sup\Big\{\Big|\frac{\partial f^i}{\partial u^j}(r, u^1, \dots, u^m)\Big| : (r, u^1, \dots, u^m) \in (r_0, 1) \times (0, M]^m\Big\} < \infty.$$

Let $(u^1, \dots, u^m) \in C^2(D \setminus \{0\}) \cap C(\overline{D})$ be a solution of (1.1) such that each u^i is n-mode, positive and of class C^n at the origin. Then, each u^i is radially symmetric and $\frac{\partial u^i}{\partial r}(|x|) < 0$ for $r = |x|$.

For scalar equations, this result was obtained in [19]. Due to the recent interest of the community for the symmetry issues for elliptic systems, we believe that the statement above is of interest. Also, it admits some interesting consequences, see for instance Theorem 1.2 below. Of course, system (1.1) includes both variational and nonvariational problems or systems of Hamiltonian type, see e.g. [7] for a wide overview. We point out, in particular, that the weakly coupled semilinear Schrödinger systems, see [14] and the references therein, which come from physically relevant situations and have recently received much attention, satisfy conditions (ii)-(iv).

For the sake of completeness, we refer the reader to [3–5] for recent partial (foliated Schwarz symmetry) symmetry results for the smooth solutions to (1.1) in rotationally invariant domains and for possibly sign-changing solutions and where the maps $r \mapsto f^i(r, s_1, \dots, s^m)$ are possibly nondecreasing and some convexity assumptions are assumed on the s_i variables.

For every $\alpha \geq 0$ and $p, q > 1$, let us set

$$R_{\alpha,p,q}(u,v) := \frac{\displaystyle\int_D (|\nabla u|^2 + |\nabla v|^2)}{\Big(\displaystyle\int_D |x|^\alpha |u|^p |v|^q\Big)^{\frac{2}{p+q}}}, \qquad \text{for any } u, v \in H^1_0(D).$$

Moreover, for each $\gamma > 0$, we shall denote by $\lceil \gamma \rceil$ the smallest integer greater than or equal to γ. Then, we have the following

Theorem 1.2. *The following facts hold.*

(I) *If $\alpha \in (0,\infty)$, $p, q \in (1,\infty)$ and (u,v) is an n-mode solution of (1.2) with $n \geq 1 + \lceil \alpha/2 \rceil$, then u, v are radially symmetric.*

(II) *For each $\alpha \in (2,\infty)$ and $p, q \in (1,\infty)$, if $n_\alpha \geq 1$ then (1.2) has a nonradial n-mode solution (u_n, v_n) for $n = 1, \dots, n_\alpha$ such that*

$$R_{\alpha,p,q}(u_1, v_1) < \cdots < R_{\alpha,p,q}(u_{n_\alpha}, v_{n_\alpha}), \tag{1.3}$$

where n_α is the greatest integer less than

$$\Big(\frac{\alpha+2}{2\alpha}\Big)^{\frac{4}{p+q-2}} \Big(\frac{\alpha-2}{\alpha}\Big)^{\frac{2\alpha}{p+q-2}} \Big(1 + \frac{\alpha}{2}\Big). \tag{1.4}$$

In particular, the following facts hold.

(i) For each $\alpha \in (2,\infty)$, if at least one of $p,q \in (1,\infty)$ is large enough, then $n_\alpha = \lceil \alpha/2 \rceil$, that is (1.2) has a nonradial n-mode solution for $n = 1,\dots,\lceil \alpha/2 \rceil$ satisfying (1.3).

(ii) For each $p,q \in (1,\infty)$ and $\ell \in \mathbb{N}$, if $\alpha \in (2,\infty)$ is large enough (1.2) has a nonradial n-mode solution (u_n,v_n) for $n = 1,\dots,\ell$ such that

$$R_{\alpha,p,q}(u_1,v_1) < \cdots < R_{\alpha,p,q}(u_\ell,v_\ell).$$

In particular, for each $p,q \in (1,\infty)$, the number of nonradial solutions of (1.2) tends to infinity as $\alpha \to \infty$.

Hence, as far as α gets large, the symmetry breaking phenomenon occurs and we can find as many n-mode positive solutions as we want. In Section 2 we shall prove Theorem 1.1, while in Section 3 we shall provide the proof of Theorem 1.2.

2. Proof of Theorem 1.1

By using the planar polar coordinates, for each $i = 1,\dots,m$, we define the function $\widetilde{u}^i : \overline{D} \to \mathbb{R}$ by setting $\widetilde{u}^i(r,\theta) := u^i(r^{1/n},\theta/n)$, for every $(r,\theta) \in \overline{D}$. Since $(u^1,\dots,u^m)$ satisfies system (1.1) and each u^i is n-mode, we can see that $\widetilde{u}^i \in C^2(D\setminus\{0\}) \cap C(\overline{D})$ and $(\widetilde{u}^1,\dots,\widetilde{u}^m)$ satisfies the system

$$\begin{cases} \Delta \widetilde{u}^i + \widetilde{f}^i(|x|,\widetilde{u}^1,\dots,\widetilde{u}^m) = 0 & \text{in } D\setminus\{0\}, \\ \widetilde{u}^i = 0 & \text{on } \partial D, \end{cases} \qquad \text{for } i = 1,\dots,m. \tag{2.1}$$

Here, $\widetilde{f}^i \in C((0,1)\times(0,\infty)^m,\mathbb{R})$ is the function defined by

$$\widetilde{f}^i(r,t^1,\dots,t^m) := n^{-2} r^{(2-2n)/n} f^i(r^{1/n},t^1,\dots,t^m),$$

for every $(r,t^1,\dots,t^m) \in (0,1)\times(0,\infty)^m$ and $\widetilde{f}^i$ satisfies

$$\text{for each } (t^1,\dots,t^m) \in (0,\infty)^m,\ r \mapsto \widetilde{f}^i(r,t^1,\dots,t^m) \text{ is nonincreasing.} \tag{2.2}$$

Indeed, from

$$\begin{aligned} &\Delta \widetilde{u}^i(r,\theta) + \widetilde{f}^i(r,\widetilde{u}^1(r,\theta),\dots,\widetilde{u}^m(r,\theta)) \\ &= \frac{1}{n^2} r^{\frac{2-2n}{n}} \left(u^i_{rr}\left(r^{\frac{1}{n}},\frac{\theta}{n}\right) + \frac{1}{r^{\frac{1}{n}}} u^i_r\left(r^{\frac{1}{n}},\frac{\theta}{n}\right) + \frac{1}{r^{\frac{2}{n}}} u^i_{\theta\theta}\left(r^{\frac{1}{n}},\frac{\theta}{n}\right) \right. \\ &\qquad \left. + f^i\left(r^{\frac{1}{n}}, u^1\left(r^{\frac{1}{n}},\frac{\theta}{n}\right),\dots,u^m\left(r^{\frac{1}{n}},\frac{\theta}{n}\right)\right)\right) = 0, \end{aligned}$$

we deduce (2.1), and we can easily see that (2.2) holds as well, in light of assumption (i). For each $\lambda \in (0,1)$, we set $\Sigma_\lambda = \{x \in D : x_1 > \lambda\}$ and we define the map $h_\lambda : \overline{\Sigma_\lambda} \to \overline{D}$ by $h_\lambda(x) = (2\lambda - x_1, x_2)$ for $x = (x_1,x_2) \in \overline{\Sigma_\lambda}$. We note that h_λ satisfies

$$|h_\lambda(x)| < |x| \quad \text{for each } \lambda \in (0,1) \text{ and } x \in \Sigma_\lambda \cup \mathrm{Int}_{\partial D}(\overline{\Sigma_\lambda} \cap \partial D). \tag{2.3}$$

Here, for a subset E of ∂D, we denote by $\mathrm{Int}_{\partial D}E$, the interior set of E with respect to the relative topology of ∂D. We set $x_\lambda = (2\lambda,0)$ for $\lambda \in (0,1)$. We can see

$$x_\lambda \in \begin{cases} \Sigma_\lambda & \text{for each } \lambda \in (0,\frac{1}{2}), \\ \partial\Sigma_\lambda & \text{for } \lambda = \frac{1}{2}, \\ \mathbb{R}^2 \setminus \overline{\Sigma_\lambda} & \text{for each } \lambda \in (\frac{1}{2},1) \end{cases} \tag{2.4}$$

and

$$h_\lambda(x_\lambda) = 0 \quad \text{for each } \lambda \in (0,1/2]. \tag{2.5}$$

For the sake of completeness, we note that $\Sigma_\lambda\setminus\{x_\lambda\}=\Sigma_\lambda$ for each $\lambda\in[\frac{1}{2},1)$ and $\overline{\Sigma_\lambda}\setminus\{x_\lambda\}=\overline{\Sigma_\lambda}$ for each $\lambda\in(\frac{1}{2},1)$. For each $i,j=1,\dots,m$, we define $v_\lambda^i\in C^2(\Sigma_\lambda\setminus\{x_\lambda\})\cap C(\overline{\Sigma_\lambda})$ and $c_\lambda^{ij}\in L^\infty(\Sigma_\lambda)$ by setting

$$v_\lambda^i(x):=\widetilde{u}^i(x)-\widetilde{u}^i(h_\lambda(x)),\qquad \text{for } x\in\overline{\Sigma_\lambda}, \tag{2.6}$$

and

$$c_\lambda^{ij}(x):=-\int_0^1\frac{\partial\widetilde{f}^i}{\partial u^j}\big(|x|,s\widetilde{u}^1(x)+(1-s)\widetilde{u}^1(h_\lambda(x)),\dots,s\widetilde{u}^m(x)+(1-s)\widetilde{u}^m(h_\lambda(x))\big)\,ds. \tag{2.7}$$

By the assumptions of Theorem 1.1, we can see that $c_\lambda^{ij}\le 0$ if $i\neq j$ for $x\in\Sigma_\lambda$ and

$$\sup_{r<\lambda<1}\sup_{x\in\Sigma_\lambda}|c_\lambda^{ij}(x)|<\infty,\qquad \text{for each } r\in(0,1) \text{ and } i,j=1,\dots,m. \tag{2.8}$$

Therefore, it holds

$$-\Delta v_\lambda^i(x)+\sum_{j=1}^m c_\lambda^{ij}(x)v_\lambda^j(x)\le 0\quad \text{for } \lambda\in(0,1),\ x\in\Sigma_\lambda\setminus\{x_\lambda\} \text{ and } i=1,\dots,m. \tag{2.9}$$

Indeed, (2.9) can be obtained as follows:

$$\begin{aligned}
0&=\Delta\widetilde{u}^i(h_\lambda(x))+\widetilde{f}^i(|h_\lambda(x)|,\widetilde{u}^1(h_\lambda(x)),\dots,\widetilde{u}^m(h_\lambda(x)))\\
&\qquad-\Delta\widetilde{u}^i(x)-\widetilde{f}^i(|x|,\widetilde{u}^1(x),\dots,\widetilde{u}^m(x))\\
&\ge-\Delta v_\lambda^i(x)+\widetilde{f}^i(|x|,\widetilde{u}^1(h_\lambda(x)),\dots,\widetilde{u}^m(h_\lambda(x)))-\widetilde{f}^i(|x|,\widetilde{u}^1(x),\dots,\widetilde{u}^m(x))\\
&=-\Delta v_\lambda^i(x)+\sum_{j=1}^m c_\lambda^{ij}(x)v_\lambda^j(x).
\end{aligned}$$

We set

$$\begin{aligned}
&A_1=\{\lambda\in[1/2,1): v_\lambda^i(x)<0 \text{ for each } x\in\Sigma_\lambda \text{ and } i\in\{1,\dots,m\}\},\\
&\mu_1=\inf_{\lambda\in A_1}\lambda.
\end{aligned} \tag{2.10}$$

We now claim that $A_1\neq\emptyset$. Let $i\in\{1,\dots,m\}$ and $\lambda\in[1/2,1)$ such that λ is sufficiently close to 1. Then we can easily see $v_\lambda^i(x)\le 0$ for $x\in\partial\Sigma_\lambda$ and $v_\lambda^i(x)<0$ for $x\in\mathrm{Int}_{\partial D}(\partial D\cap\partial\Sigma_\lambda)$ from (2.3). Since $|\Sigma_\lambda|\ll 1$ and (2.9) holds, by [2, Corollary 14.1], we have $v_\lambda^i\le 0$ on $\overline{\Sigma_\lambda}$. By

$$-\Delta v_\lambda^i(x)+c_\lambda^{ii}(x)v_\lambda^i(x)\le-\sum_{j\neq i}^m c_\lambda^{ij}(x)v_\lambda^j(x)\le 0\quad \text{in } \Sigma_\lambda\setminus\{x_\lambda\} \tag{2.11}$$

and the strong maximum principle, we have $v_\lambda^i<0$ in Σ_λ. Since i is any element of $\{1,\dots,m\}$, we have shown $\lambda\in A_1$, which proves the claim.

We now claim that $\mu_1=1/2\in A_1$. Let $i\in\{1,\dots,m\}$. We have $v_{\mu_1}^i(x)\le 0$ for $x\in\Sigma_{\mu_1}$. Since (2.11) holds with $\lambda=\mu_1$ and $v_{\mu_1}^i(x)<0$ for $x\in\mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_1}\cap\partial D)$ from (2.3), by the strong maximum principle, we have $v_{\mu_1}^i(x)<0$ for $x\in\Sigma_{\mu_1}$. Since i is an arbitrary element of $\{1,\dots,m\}$, we have $\mu_1\in A_1$. We will show $\mu_1=1/2$. Suppose not, namely $\mu_1>1/2$. Again let $i\in\{1,\dots,m\}$. Let G be an open set such that $\overline{G}\subset\Sigma_{\mu_1}$ and $|\Sigma_{\mu_1}\setminus\overline{G}|\ll 1$. We have $\max_{x\in\overline{G}}v_{\mu_1}^i(x)<0$. Let $0<\varepsilon\ll 1$. Then we have $\max_{x\in\overline{G}}v_{\mu_1-\varepsilon}^i(x)<0$ and $|\Sigma_{\mu_1-\varepsilon}\setminus\overline{G}|\ll 1$. Since (2.9) holds with $\lambda=\mu_1-\varepsilon$ and $v_{\mu_1-\varepsilon}^i(x)\le 0$ for $x\in\partial(\Sigma_{\mu_1-\varepsilon}\setminus\overline{G})$, we have $v_{\mu_1-\varepsilon}^i(x)\le 0$ for $x\in\Sigma_{\mu_1-\varepsilon}$ by [2, Corollary 14.1]. From $v_{\mu_1-\varepsilon}^i(x)<0$ for $x\in(\mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_1-\varepsilon}\cap\partial D))\cup\partial G$ and the strong maximum principle, we have $v_{\mu_1-\varepsilon}^i(x)<0$ for $x\in\Sigma_{\mu_1-\varepsilon}$. Since i is an arbitrary element of $\{1,\dots,m\}$, we have $\mu_1-\varepsilon\in A_1$. This is a contradiction. Hence $\mu_1=1/2\in A_1$.

We now set

$$\begin{aligned} &A_2 := \{\lambda \in (0,1/2) : \ v^i_\lambda(x) < 0 \text{ for each } x \in \Sigma_\lambda \text{ and } i \in \{1,\dots,m\}\}, \\ &\mu_2 := \inf_{\lambda \in A_2} \lambda. \end{aligned} \tag{2.12}$$

We now claim that $A_2 \neq \emptyset$. Let $i \in \{1,\dots,m\}$. We note that $x_{1/2} = (1,0)$. Let G be an open set such that $\overline{G} \subset \Sigma_{1/2}$ and $|\Sigma_{1/2} \backslash G| \ll 1$. From $1/2 \in A_1$ and $\overline{G} \subset \Sigma_{1/2}$, we have $\max_{x\in\overline{G}} v^i_{1/2}(x) < 0$. Let $\lambda \in (0,1/2)$ such that λ is sufficiently close to $1/2$. We note $|\Sigma_\lambda \setminus \overline{G}| \ll 1$ and x_λ is close to $(1,0)$. We choose a sufficiently small open neighborhood U of x_λ with $\overline{U} \subset \Sigma_\lambda$, and we set $H = G \cup U$. Then we have $v^i_\lambda(x) < 0$ for $x \in \overline{H}$, $v^i_\lambda(x) \le 0$ for $x \in \partial\Sigma_\lambda \cup \partial H$ and $|\Sigma_\lambda \setminus \overline{H}| \ll 1$. Since (2.9) holds on $\Sigma_\lambda \setminus \overline{H}$, by [2, Corollary 14.1], we have $v^i_\lambda \le 0$ on Σ_λ. From (2.11) and the strong maximum principle, we have $v^i_\lambda < 0$ on Σ_λ. Since i is an arbitrary element of $\{1,\dots,m\}$, we have shown $\lambda \in A_2$.
Recalling that u is of class C^n at the origin, arguing exactly as in [19, Lemma 4] we get

$$\frac{\partial(\widetilde{u}^i \circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = 0 \quad \text{for each } i = 1,\dots,m. \tag{2.13}$$

We now claim that $\mu_2 = 0$. Suppose not. Let $i \in \{1,\dots,m\}$. Then we have $\mu_2 \in (0,1/2)$ by the previous claim and we can see $v^i_{\mu_2} \le 0$ on $\overline{\Sigma_{\mu_2}}$. We will show $v^i_{\mu_2} < 0$ on $\Sigma_{\mu_2} \setminus \{x_{\mu_2}\}$. We have $v^i_{\mu_2}(x) < 0$ for $x \in \mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_2} \cap \partial D)$ from (2.3). By (2.11) with $\lambda = \mu_2$ and the strong maximum principle, we have $v^i_{\mu_2} < 0$ on $\Sigma_{\mu_2} \setminus \{x_{\mu_2}\}$. Next, we will show $v^i_{\mu_2}(x_{\mu_2}) < 0$. Suppose $v^i_{\mu_2}(x_{\mu_2}) < 0$ does not hold, i.e., $v^i_{\mu_2}(x_{\mu_2}) = 0$. Let $\nu_1 = (-1,0)$ and $\nu_2 = (1,0)$. From (2.13), we have

$$\begin{aligned} \frac{\partial v^i_{\mu_2}}{\partial \nu_1}(x_{\mu_2}) &= -\frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}) + \frac{\partial(\widetilde{u}^i \circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = -\frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}), \\ \frac{\partial v^i_{\mu_2}}{\partial \nu_2}(x_{\mu_2}) &= \frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}) - \frac{\partial(\widetilde{u}^i \circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = \frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}). \end{aligned}$$

By Hopf's lemma, we obtain

$$-\frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}) < 0 \quad \text{and} \quad \frac{\partial \widetilde{u}^i}{\partial x_1}(x_{\mu_2}) < 0,$$

which is a contradiction. So we have shown $v^i_{\mu_2}(x_{\mu_2}) < 0$. Thus we have $v^i_{\mu_2} < 0$ on Σ_{μ_2}. Since i is an arbitrary element of $\{1,\dots,m\}$, we have $\mu_2 \in A_2$. Again let $i \in \{1,\dots,m\}$. We choose an open set G such that $\overline{G} \subset \Sigma_{\mu_2}$ and $|\Sigma_{\mu_2} \setminus \overline{G}| \ll 1$. We have $\max_{\overline{G}} v^i_{\mu_2} < 0$. Let $0 < \varepsilon \ll 1$. Then we have $|\Sigma_{\mu_2-\varepsilon} \setminus \overline{G}| \ll 1$ and $\max_{\overline{G}} v^i_{\mu_2-\varepsilon} < 0$. Since (2.9) holds with $\lambda = \mu_2 - \varepsilon$, by [2, Corollary 14.1], we have $v^i_{\mu_2-\varepsilon}(x) \le 0$ for $x \in \Sigma_{\mu_2-\varepsilon} \setminus \overline{G}$. By (2.11) with $\lambda = \mu_2 - \varepsilon$ and the strong maximum principle, we have $v^i_{\mu_2-\varepsilon}(x) < 0$ for $x \in \Sigma_{\mu_2-\varepsilon} \setminus \overline{G}$. Hence we have shown $v^i_{\mu_2-\varepsilon}(x) < 0$ for $x \in \Sigma_{\mu_2-\varepsilon}$. Since i is an arbitrary element of $\{1,\dots,m\}$, we have $\mu_2 - \varepsilon \in A_2$, which is a contradiction. Therefore we obtain $\mu_2 = 0$.
We can finally conclude the proof of Theorem 1.1. Let $i \in \{1,\dots,m\}$. By the conclusions above, we can infer that $\widetilde{u}^i$ is radially symmetric and $\frac{\partial \widetilde{u}^i}{\partial r}(|x|) < 0$ for $r = |x| \in (0,1)$. From the definition of $\widetilde{u}^i$, we can find u^i is also radially symmetric and $\frac{\partial u^i}{\partial r}(|x|) < 0$.

3. Proof of Theorem 1.2

Let us first prove assertion ($\mathbb{I}$) of Theorem 1.2. Assume that (u,v) is an n-mode solution to system (1.2) such that $n \ge 1 + \lceil \alpha/2 \rceil$. Then, we may choose $\hat{n}, m \in \mathbb{N}$ such that $m/\hat{n} \in \mathbb{N}$,

$(\alpha+2)\hat{n} \le 2n < 2(\alpha+2)\hat{n}$ and

$$m > \max\Big\{\frac{n\hat{n}}{(\alpha+2)\hat{n}-n}, \frac{2n}{\alpha+2}\Big\}.$$

Setting $\hat{u}(r,\theta) := u(r^{m/n}, m\theta/n)$ and $\hat{v}(r,\theta) := v(r^{m/n}, m\theta/n)$, it is readily seen that $\hat{u}$ and $\hat{v}$ are both $\hat{m} = m/\hat{n}$-mode and solve, in $D \setminus \{0\}$, the system

$$\begin{cases} \Delta\hat{u} + \dfrac{2pm^2}{(p+q)n^2}|x|^{\frac{m(\alpha+2)-2n}{n}}\hat{u}^{p-1}\hat{v}^{q} = 0 & \text{in } D\setminus\{0\}, \\ \Delta\hat{v} + \dfrac{2pm^2}{(p+q)n^2}|x|^{\frac{m(\alpha+2)-2n}{n}}\hat{u}^{p}\hat{v}^{q-1} = 0 & \text{in } D\setminus\{0\}, \\ \hat{u} > 0,\ \hat{v} > 0 & \text{in } D\setminus\{0\}, \\ \hat{u} = \hat{v} = 0 & \text{on } \partial D. \end{cases}$$

We need to show that $(\hat{u},\hat{v})$ is a solution of the corresponding system on D, namely

$$\begin{cases} \Delta\hat{u} + \dfrac{2pm^2}{(p+q)n^2}|x|^{\frac{m(\alpha+2)-2n}{n}}\hat{u}^{p-1}\hat{v}^{q} = 0 & \text{in } D, \\ \Delta\hat{v} + \dfrac{2pm^2}{(p+q)n^2}|x|^{\frac{m(\alpha+2)-2n}{n}}\hat{u}^{p}\hat{v}^{q-1} = 0 & \text{in } D. \end{cases} \tag{3.1}$$

To this aim, let $\varphi \in C_c^\infty(D)$ a function and let $\varepsilon \in (0,1)$. Then, if D_ε denotes the ball centered at zero with radius ε, we get

$$\begin{aligned} 0 &= \int_{D\setminus D_\varepsilon} \Delta\hat{u}\varphi + \frac{2pm^2}{(p+q)n^2}\int_{D\setminus D_\varepsilon} |x|^{\frac{m(\alpha+2)-2n}{n}}\hat{u}^{p-1}\hat{v}^{q}\varphi \\ &= -\int_{\partial D_\varepsilon}\frac{\partial\hat{u}}{\partial r}\varphi dS - \int_{D\setminus D_\varepsilon}\nabla\hat{u}\nabla\varphi + \frac{2pm^2}{(p+q)n^2}\int_{D\setminus D_\varepsilon} |x|^{\frac{m(\alpha+2)-2n}{n}}\hat{u}^{p-1}\hat{v}^{q}\varphi, \\ 0 &= \int_{D\setminus D_\varepsilon} \Delta\hat{v}\varphi + \frac{2pm^2}{(p+q)n^2}\int_{D\setminus D_\varepsilon} |x|^{\frac{m(\alpha+2)-2n}{n}}\hat{u}^{p}\hat{v}^{q-1}\varphi \\ &= -\int_{\partial D_\varepsilon}\frac{\partial\hat{v}}{\partial r}\varphi dS - \int_{D\setminus D_\varepsilon}\nabla\hat{v}\nabla\varphi + \frac{2pm^2}{(p+q)n^2}\int_{D\setminus D_\varepsilon} |x|^{\frac{m(\alpha+2)-2n}{n}}\hat{u}^{p}\hat{v}^{q-1}\varphi. \end{aligned}$$

Since also $u \in C^1(\overline{D})$, the functions $\frac{\partial u}{\partial r}(r,\theta)$, $\frac{1}{r}\frac{\partial u}{\partial\theta}(r,\theta)$, $\frac{\partial\varphi}{\partial r}(r,\theta)$, $\frac{1}{r}\frac{\partial\varphi}{\partial\theta}(r,\theta)$ are bounded on D. From $\bar{u}(r,\theta) = u(r^{\frac{m}{n}}, \frac{m}{n}\theta)$, there exists some positive constant C such that

$$\Big|\frac{\partial\bar{u}}{\partial r}(r,\theta)\Big| \le Cr^{\frac{m}{n}-1}, \quad \Big|\frac{\partial\bar{u}}{\partial\theta}(r,\theta)\Big| \le Cr,$$

for each $(r,\theta) \in D$. Hence, we have

$$\Big|\int_{|x|=\varepsilon}\frac{\partial\bar{u}}{\partial r}\varphi\, dS\Big| \le C\varepsilon^{\frac{m}{n}},$$

$$\Big|\int_{\{\varepsilon<|x|<1\}}\nabla\bar{u}\,\nabla\varphi - \int_D\nabla\bar{u}\,\nabla\varphi\Big| \le C\varepsilon^{\frac{m}{n}+1} + C\varepsilon^2.$$

Hence, letting $\varepsilon \to +0$, we conclude that $(\hat{u},\hat{v})$ is a weak (and hence a strong) solution to (3.1). Since $(m(\alpha+2)-2n)/n \ge \hat{m}-1$, it follows that $\hat{u},\hat{v} \in C^{\hat{m}}(\overline{D})$. Furthermore, since $(\alpha+2)\hat{n} \le 2n$ the map $r \mapsto r^{2-2\hat{m}+(m(\alpha+2)-2n)/n}$ is nonincreasing. In turn, by applying Theorem 1.1, it follows that $\hat{u}$ and $\hat{v}$ are radially symmetric and hence u and v are radially symmetric, concluding the first part of the proof.

We now come to the proof of assertion (III). We set $H_n = \{u \in H_0^1(D) : u \text{ is } n\text{-mode}\}$ for all $n \in \mathbb{N}$ and $H_\infty = \{u \in H_0^1(D) : u \text{ is radially symmetric}\}$. For any $p, q > 1$, $\alpha \geq 0$ and $n \in \mathbb{N} \cup \{\infty\}$, set

$$S_{\alpha,p,q,n} = \inf\{R_{\alpha,p,q}(u,v) : u, v \in H_n \setminus \{0\}\}.$$

From the proof of [23, Proposition 2.5], we can find

$$S_{\alpha,p,q,\infty} \geq S_{0,p,q,1}\Big(\frac{\alpha+2}{2}\Big)^{1+\frac{2}{p+q}}. \tag{3.2}$$

Next, let φ be any element of $C_0^\infty(D)$. Since we can consider $\varphi \in C_0^\infty(\mathbb{R}^2)$ by the trivial extension, we can define $\varphi_\alpha \in C_0^\infty(D)$ by $\varphi_\alpha((x_1,x_2)) = \varphi(\alpha(x_1 - (1 - 1/\alpha)), \alpha x_2)$ for $(x_1, x_2) \in D$. We set $D_1 = D$ and

$$D_n = \{(r,\theta) : 0 < r < 1, -\pi/n < \theta < \pi/n\} \quad \text{for } n \in \mathbb{N} \setminus \{1\}.$$

For each $n \in \mathbb{N}$ and $\alpha > 0$ with $\operatorname{supp}\varphi_\alpha \subset D_n$, we will show

$$S_{\alpha,p,q,n} \leq S_{0,p,q,1} n^{1-\frac{2}{p+q}} \alpha^{\frac{4}{p+q}} \Big(\frac{\alpha}{\alpha-2}\Big)^{\frac{2\alpha}{p+q}}. \tag{3.3}$$

We define $P_n : D \to D$ by $P_n(r,\theta) = (r, \theta + 2\pi/n)$ for $(r,\theta) \in [0,1) \times \mathbb{R}$. We set $\widetilde{\varphi}(x) = \varphi_\alpha(x) + \varphi_\alpha(P_n(x)) + \cdots + \varphi_\alpha(P_n^{n-1}(x))$ for $x \in D$. Since we have

$$\int_D |\nabla \varphi_\alpha|^2 = \int_D |\nabla \varphi|^2$$

and

$$\int_D |x|^\alpha |\varphi_\alpha|^p |\varphi_\alpha|^q \geq \alpha^{-2}\left(1 - \frac{2}{\alpha}\right)^\alpha \int_D |\varphi|^p|\varphi|^q,$$

we obtain

$$S_{\alpha,p,q,n} \leq R_{\alpha,p,q}(\widetilde{\varphi},\widetilde{\varphi}) \leq \frac{n\displaystyle\int_D (|\nabla\varphi|^2 + |\nabla\varphi|^2)}{\Big(n\alpha^{-2}\Big(1-\frac{2}{\alpha}\Big)^\alpha \displaystyle\int_D |\varphi|^p|\varphi|^q\Big)^{\frac{2}{p+q}}}.$$

Since $\varphi \in C_0^\infty(D)$ is arbitrary, we have shown (3.3). From (3.2) and (3.3), we can see that $n \leq n_\alpha$ is a sufficient condition for $S_{\alpha,p,q,n} < S_{\alpha,p,q,\infty}$. We will show that if $n > 1$ and $S_{\alpha,p,q,n} < S_{\alpha,p,q,\infty}$ then $S_{\alpha,p,q,1} < \cdots < S_{\alpha,p,q,n}$. Let $n > 1$ and $S_{\alpha,p,q,n} < S_{\alpha,p,q,\infty}$. We can choose $u, v \in H_n \setminus \{0\}$ such that $R_{\alpha,p,q}(u,v) = S_{\alpha,p,q,n}$ and $u, v \geq 0$. We note that $u, v \notin H_\infty$ and (u,v) is a positive solution of (1.2). Let $m \in \{1, \ldots, n-1\}$. We define $\bar{u}, \bar{v} \in H_m$ by $\bar{u}(r,\theta) = u(r, m\theta/n)$ and $\bar{v}(r,\theta) = v(r, m\theta/n)$ for $(r,\theta) \in [0,1) \times \mathbb{R}$. Since we can see

$$\int_D |x|^\alpha |\bar{u}|^p|\bar{v}|^q = \int_D |x|^\alpha |u|^p |v|^q,$$

$$\int_D |\nabla \bar{u}|^2 = \int_0^{2\pi}\int_0^1 \Big(\left|\frac{\partial u}{\partial r}\right|^2 + \frac{m^2}{n^2 r^2}\left|\frac{\partial u}{\partial \theta}\right|^2\Big) r\,dr d\theta < \int_D |\nabla u|^2$$

and $\int_D |\nabla \bar{v}|^2 < \int_D |\nabla v|^2$, we have

$$S_{\alpha,p,q,m} \leq R_{\alpha,p,q}(\bar{u},\bar{v}) < R_{\alpha,p,q}(u,v) = S_{\alpha,p,q,n}.$$

By a similar argument, we conclude that $S_{\alpha,p,q,1} < \cdots < S_{\alpha,p,q,n}$. Hence we infer that if $S_{\alpha,p,q,n} < S_{\alpha,p,q,\infty}$, then for each $\ell = 1, \ldots, n$, there exists a nonradial positive solution $(u_\ell, v_\ell) \in H_\ell \times H_\ell$ of (1.2) satisfying $R_{\alpha,p,q}(u_\ell, v_\ell) = S_{\alpha,p,q,\ell}$. We set the number in (1.4) as $\eta(\alpha,p,q)$. For a fixed $\alpha \in (2,\infty)$, we have $\eta(\alpha,p,q) \to 1 + \alpha/2$ as $p + q \to \infty$, which yields (i). For a fixed $p, q \in (1,\infty)$, we have $\eta(\alpha,p,q) \to \infty$ as $\alpha \to \infty$, yielding (ii). Hence, we finish the proof of Theorem 1.2.

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